Stability of stochastic functional differential systems using degenerate Lyapunov functionals and applications

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\textbf{ABSTRACT}

Motivated by the seminal work of Dupire (2009) on functional Itô formulas, this work investigates asymptotic properties of systems represented by stochastic functional differential equations (SFDEs). Stability of general delay-dependent SFDEs is investigated using degenerate Lyapunov functionals, which are only positive semi-definite rather than positive definite as used in the classical work. This paper first establishes boundedness and regularity of SFDEs by using degenerate Lyapunov functionals. Then moment and almost sure exponential stabilities are obtained based on degenerate Lyapunov functionals and the semi-martingale convergence theorem. As an application of the stability criteria, consentability of stochastic multi-agent systems with nonlinear dynamics is studied.

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\textbf{1. Introduction}

In his seminal work (Dupire, 2009), Dupire extended the Itô formula to a functional setting by using a pathwise functional derivative. This work has substantially eased the difficulties in finding solutions for non-Markovian processes due to time delays, that have defied bona fide operators and functional Itô formulas in the past. The current paper aims to use the newly developed functional Itô formula to examine stability and related issues of stochastic functional differential systems.

Many real systems, including population biology, epidemiology, economics, neural networks, and control of mechanical and electrical systems, inevitably involve delays, leading to delay differential equations (DDEs) and more general functional differential equations (FDEs) (Hale & Lunel, 1993; Kolmanovskii & Myshkis, 1999; Luo, Gong, & Jia, 2017). Some of their important properties have been established (Bejarano & Zheng, 2014; Bresch-Pietri & Krstic, 2014).

Although small delays may occasionally enhance stability in some systems (Yu, Chen, Cao, & Ren, 2013), delays are typically a source of instability, poor performance, and difficulty for analysis and control design. For linear time invariant systems, eigenvalues of system matrices may be used to deduce stability (Hale & Lunel, 1993). More complex functional equations are often treated by using Lyapunov functional and Razumikhin methods (Hale & Lunel, 1993; Kolmanovskii & Nosov, 1986). Combined with stochastic disturbances, delay systems become stochastic delay differential equations (SDDEs) and stochastic functional differential equations (SFDEs) (Mao, 1997; Mohammed, 1986), resulting in the consideration of moment stability, almost sure stability, and stability in probability. It is extremely difficult to establish necessary and sufficient conditions of mean square and almost sure stabilities for SFDEs. At present, Razumikhin methods and Lyapunov functionals are two main tools for establishing sufficient stability conditions. The stochastic versions of Razumikhin methods were developed in Janković, Randjelović, and Jovanović (2009), Mao (1996), Wu and Hu (2012) and Wu, Yin, and Wang (2015) for studying moment asymptotic stability. However, unlike its deterministic counterpart, the stochastic Razumikhin methods have limited success (Hale & Lunel, 1993; Teel, 1998).
While sufficient conditions on delay-independent stability of SFDEs are often obtained by using Lyapunov functions, studies on delay-dependent stability mainly employ Lyapunov functional and linear matrix inequalities (LMIs). Mao (1997) established the delay-independent ρ-th-moment stability by using Lyapunov functions and obtained the almost sure stability from the moment exponential stability and a linear growth condition. Huang and Mao (2009a) studied the mean square exponential stability of neutral SDDEs by using a Lyapunov functional, and certain stability conditions were given in terms of LMIs. Caraballo, Real, and Taniguchi (2007) studied the almost sure exponential stability and ultimate boundedness of solutions for a class of neutral stochastic semilinear partial delay differential equations. Rakkianpan, Balasubramaniam, and Lakshmanan (2008) used Lyapunov functionals to derive LMI-type stability conditions for uncertain stochastic neural networks. Gershon, Shaked, and Berman (2007) investigated $H_\infty$ state-feedback control of stochastic delay systems by Lyapunov functions and LMIs. Shaiel (2013) introduced different Lyapunov functionals to examine stochastic stabilities of different SFDEs. These results were given under specific Lyapunov functionals. A fundamental stability theory related to Lyapunov functions has not been developed.

Regarding delayed feedback stabilization problems of stochastic systems, Verriest and Florchinger (1995) gave delay-independent conditions on delayed feedback stabilization of linear SDDEs by Riccati-type equations in which multi-matrices must be determined. Huang and Mao (2009b) and Mao (2002) obtained delay-dependent exponential stability conditions for linear SDDEs in terms of LMIs. Karimi (2011) investigated robust mode-dependent delayed state feedback control for a class of uncertain delay systems in terms of LMIs.

All of the above references are based on positive definitive Lyapunov functionals. In this paper, using degenerate Lyapunov functionals, we develop several fundamental stability theorems, including moment asymptotic stability, exponential stability, and almost sure exponential stability. Degenerate Lyapunov functionals being positive semi-definite were well investigated in Kolmanovskii and Myshkis (1999) for deterministic neutral FDEs. Some specific degenerate Lyapunov functionals were also applied to examine the moment stability of stochastic functional differential equations with global Lipschitz conditions (Kolmanovskii & Myshkis, 1999; Kolmanovskii & Shaikhet, 1997; Shaikhet, 2013). Due to lack of bona fide operators and Itô formulas, whether degenerate Lyapunov functionals can be used to examine stability of SFDEs with nonlinear dynamics, nonlinear growth, and non-Lipschitz coefficients remains an open issue.

This paper fills in the gap. Our results reveal that appropriate degenerate Lyapunov functionals can be used to simplify stability analysis and control design. The results are then applied to multi-agent consensus and consensus problems, which are motivated by highway autonomous vehicles, unmanned aerial vehicles, team robots, among other applications (Cheng, Hou, & Tan, 2014; Li, Fu, Xie, & Zhang, 2011; Zong, Li, Yin, Wang, & Zhang, 2017; Zong, Li, & Zhang, 2017). While nonlinear multi-agent systems have attracted much attention (Strogatz, 2001; Zhu, Xie, Han, Meng, & Teo, 2017), stochastic multi-agent consensus and consensus with nonlinear dynamics, even for the delay-free case, are still not resolved. The existing LMI-type conditions are not explicit for consensus stability. The results of this paper resolve the issue. Assuming that the nonlinear terms satisfy a Lipschitz condition with uncertainty, we give sufficient stochastic consensus stability conditions, and demonstrate explicit that the multi-agent consensus depends on the delay in feedback, noise intensities, balanced graphs, and the Lipschitz constant of the nonlinear dynamics.

The rest of the paper is structured as follows. Section 2 introduces the notation and develops the functional Itô formula for SFDEs. Section 3 establishes the fundamental degenerate Lyapunov functional theorem to produce the regularity and moment boundedness of SFDEs. Section 4 presents asymptotic moment estimates, moment stability, and almost sure stability based on the semi-martingale convergence theorem. In particular, mean square stability and almost sure stability are established for stochastic systems represented by semi-linear SFDEs. Section 5 applies the theoretical results to examine stochastic constancy of multi-agent systems with nonlinear dynamics. Section 6 concludes the paper with some remarks.

2. Functional Itô formula:

We work with the $n$-dimensional Euclidean space $\mathbb{R}^n$ equipped with the Euclidean norm $||\cdot||$. For a vector or a matrix $A$, its transpose is denoted by $A^\top$. For a matrix $A$, denote its trace norm by $|A| = \sqrt{\text{trace}(A^\top A)}$. For a symmetric matrix $A$ with real entries, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the largest and smallest eigenvalues, respectively. For two matrices $A$ and $B$, $A \otimes B$ denotes their Kronecker product. Use $a \vee b$ to denote $\max(a, b)$ and $a \wedge b$ to denote $\min(a, b)$. For $\tau > 0$, denote by $C([-\tau, 0]; \mathbb{R}^n)$ the family of all $\mathbb{R}^n$-valued continuous functions on the interval $[-\tau, 0]$ with the norm $||\phi||_C = \sup_{t \in [-\tau, 0]}|\phi(t)|$. Let $D([-\tau, 0]; \mathbb{R}^n)$ be the space of $\mathbb{R}^n$-valued functions on $[-\tau, 0]$ that are right continuous and have left-hand limits endowed with the Skorohod topology. The metric in $D([-\tau, 0]; \mathbb{R}^n)$ is given by $d_0(x, y) = \inf_{\eta \in \mathcal{E}} \sup_{t \in [-\tau, 0]}|x(t) - y(t)| < \epsilon$ for some $\delta > 0$, where $\mathcal{E}$ is the space of the continuous increasing functions from $[-\tau, 0]$ onto $[-\tau, 0]$. Under this metric, $D$ becomes a complete and separable metric space (Kushner, 1984). We work with $(\Omega, \mathcal{F}, \mathbb{P})$, a complete probability space with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Let $w(t) = (w_1(t), w_2(t), \ldots, w_d(t))$ be a $d$-dimensional standard Brownian motion. Consider the following system given by a stochastic functional differential equation (SFDE)

$$dy(t) = f(y, t)dt + g(y, t)dw(t),$$

where $y_t = \{y(t + \theta) : \theta \in [-\tau, 0]\}, f : C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ and $g : C([-\tau, 0], \mathbb{R}^d) \times \Omega \rightarrow \mathbb{R}^{nd}$ satisfy the following condition throughout the paper.

Assumption 2.1. (Local Lipschitz Condition). For each $j > 0$, there exists a constant $C_j > 0$ such that $|f(j, \psi, t) - f(j, \phi, t)| \leq C_j|\psi - \phi|$ for all $t \geq 0$ and $\psi, \phi \in C([-\tau, 0]; \mathbb{R}^n)$ with $|\psi(t)| \leq |\phi(t)| < j$.

We now develop the functional Itô formula to the semi-martingale $y$, determined by $(1)$. For each $\psi \in D([-\tau, 0], \mathbb{R}^n)$, $\psi_j(\theta) = \psi(t + \theta)$, $\theta \in [-\tau, 0]$. Motivated (Cont & Fournié, 2013; Dupire, 2009), we define the horizontal extension $\delta \psi$ of $\psi$: $\delta \psi(\theta) = \psi(\theta + \delta), \theta \in [-\tau, -\delta]$, and $\delta \psi(\theta) = \psi(0), \theta \in [-\delta, 0]$, and the vertical perturbation $\psi_j$ of $\psi$: $\psi_j(\theta) = \psi(\theta), \theta \in [-\tau, 0]$, and $\psi_j(0) = \psi(0) + v$, $v \in \mathbb{R}^n$. Let $[e_i, i = 1, \ldots, n]$ denote the canonical basis in $\mathbb{R}^n$. Then for $\psi \in D([-\tau, 0])$, we define

$$D\psi \psi(t) := \lim_{\delta \rightarrow 0^+} \frac{V(\delta \psi, t + \delta) - V(\psi, t)}{\delta},$$

$$\nabla \psi \psi(t) := \frac{\partial \psi(\psi, t)}{\partial \psi},$$

and the linear terms satisfy a Lipschitz condition with uncertainty, we give sufficient stochastic consensus stability conditions, and demonstrate explicit that the multi-agent consensus depends on the delay in feedback, noise intensities, balanced graphs, and the Lipschitz constant of the nonlinear dynamics.

The rest of the paper is structured as follows. Section 2 introduces the notation and develops the functional Itô formula for SFDEs. Section 3 establishes the fundamental degenerate Lyapunov functional theorem to produce the regularity and moment boundedness of SFDEs. Section 4 presents asymptotic moment estimates, moment stability, and almost sure stability based on the semi-martingale convergence theorem. In particular, mean square stability and almost sure stability are established for stochastic systems represented by semi-linear SFDEs. Section 5 applies the theoretical results to examine stochastic constancy of multi-agent systems with nonlinear dynamics. Section 6 concludes the paper with some remarks.
Remark 2.1. If $V(\varphi, t) = U(\varphi(0), t)$, where $U \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$, then $\nabla V(\varphi, t) = \frac{1}{2} U'(\varphi(0), t), \nabla \nabla V(\varphi, t) = \frac{1}{2} U''(\varphi(0), t)$, and $\nabla \nabla \nabla V(\varphi, t) = \frac{1}{2} U'''(\varphi(0), t)$. If $V(\varphi, t) = \int_0^t |\varphi(\tau)|^2 \, d\tau$, then

$$\frac{d}{dt} \frac{d}{dt} \cdots \frac{d}{dt} V(\varphi, t) = [\cdots [\cdots [ \frac{1}{2} U'' \circ (\cdots \circ (\cdots \circ \varphi(0))) \cdots ] \cdots ] \cdots ].$$

Remark 2.2. The functional Itô formula of Cont and Fournié (2013) and Dupire (2009) for the path-dependent stochastic systems enlarges the domains applicable for the Itô formula. In this paper, the new functional Itô formula (3) allows a constructive asymptotic analysis of SFDEs, provides a procedure to compute $\mathcal{L}V$ for general functional $V(\varphi)$, and forms the foundation of the fundamental stability criteria (Theorems 4.1–4.3 below) which also weakens the stability conditions in Kolmanovskii and Myshkis (1999, Theorem 2.1, p. 389) (see Remark 4.2). It is well known that for stochastic systems without delay, Lyapunov function play an important role in stability analysis, where conditions on $\mathcal{L}V$ for the function $V(\varphi)$ under the classical Itô formula are required (see Mao, 1997). However, for the general functional $V(\varphi)$, the classical Itô formula cannot be used to compute $\mathcal{L}V$ and the fundamental Lyapunov functional method based on $\mathcal{L}V$ cannot be established under the classical Itô formula.

3. Regularity and moment boundedness

Theorem 3.1. Assume that there exists a functional $V \in C^{2,1}(D([-\tau, 0]; \mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+), \eta(t) \geq 0$ with $\int_0^t 0 \eta(s) \, ds < \infty \text{ for any } t \in \mathbb{R}_+$ and some constants $c_0 > c_1 \geq 0$, $p > 0, 0 \geq \beta \geq 0, \lambda \in \mathbb{R}$ such that

(I) $c_0 |\varphi(0)|^p - c_1 \int_0^t |\varphi(\tau)|^p \, d\mu_1(\tau) \leq V(\varphi, t)$;

(II) $\mathcal{L}V(\varphi, t) \leq \eta(t) - U_1(|\varphi(0)|) + \int_0^t U_2(|\varphi(s)|) \, ds - \alpha U_3(|\varphi(s)|) + \frac{1}{2} U_4(|\varphi(s)|) d\mu_2(\tau)$;

(III) $U_2(x) - U_1(x) \leq -\lambda |x|^p$, for all $x \in \mathbb{R}^n$,

where $\{\mu_k\}_{k=1}^\infty$ are probability measures defined on $[-\tau, 0]$. Then $SDDE (1)$ admits a unique global solution for any initial data $y_0 \in C([-\tau, 0]; \mathbb{R}^n)$. Especially, if $\int_0^\infty \eta(s) \, ds < \infty$ and $\lambda \geq 0$, then the solution to $SDDE (1)$ is bounded in the sense of $\text{pth}$ moment, that is,

$$\sup_{t \geq 0} \mathbb{E} \|y(t)\|^p < \infty.$$

Remark 3.1. The conditions in Theorem 3.1 extend the classical Lyapunov theory for stochastic systems without delay. In fact, if

$$\tau = 0, \{\mu_k\}_{k=1}^\infty \text{ become the Dirac measure } \delta_0 \text{ and the functional } V(\varphi, t) \text{ is a certain function } V(x, t), \text{ then condition (I) reduces to } (c_0 - c_1)x^p \leq V(x, t), \text{ which is frequently used in moment stability analysis, and which justifies the requirement } c_0 > c_1. \text{ In the non-delay case, conditions (II) and (III) become } \mathbb{E} \|y(t)\|^p \leq \eta(t) - \lambda |y|^p, \text{ which is a classical condition in the Lyapunov asymptotic analysis of stochastic systems (Mao, 1997). Intuitively, conditions (II)–(III) imply that the functional can grow since } \lambda \text{ can be negative. But condition (I) is added to avoid exploding in finite time. In fact, if } \eta(t) = 0, \text{ the growth rate is no more than the exponential rate, see (9) below. However, if } \lambda > 0, \text{ then conditions (II)–(III) imply that the functional cannot grow forever. In fact, if } \eta(t) = 0, \text{ the overall solution tends to be bounded. Such conditions are often required for Lyapunov function stability of SFDEs (see Luo, Mao, & Shen, 2006; Song & Shen, 2013). Hence, the lower bound condition (I) is not required if } \lambda > 0 \text{ and } \mathbb{E} \eta(t) dt \leq \infty. \text{ More details will be given in Theorem 3.2.}$

Also, we require $V(\cdot, t) \geq 0, \text{ but not } c_0 |\varphi(0)|^p - c_1 \int_0^t |\varphi(\tau)|^p \, d\mu_1(\tau) \geq 0. \text{ For the case } d\mu_1(\tau) = ds, \tau = 1, |\varphi(s)| = (2 + s)^{-1/2}, c_0 = 0.11 \text{ and } c_1 = 0.1, \text{ if } c_0 |\varphi(0)|^p - c_1 \int_0^t |\varphi(\tau)|^p \, d\mu_1(\tau) < 0. \text{ This differs from the classical Lyapunov functionals where } c_0 |\varphi(0)|^p \leq V(\varphi, t) \text{ is required (Huang & Mao, 2009b). Hence, the functional in this paper is a degenerate Lyapunov functional. Sections 4.2 and 4.3 give some specific examples of Lyapunov functionals for the stability analysis.}$

Proof. For any given initial data $y_0 \in C([-\tau, 0], \mathbb{R}^n), \text{ by Assumption 2.1, there exists a unique maximal local strong solution } y(t) \text{ to (1) on } t \in [-\tau, \rho_0], \text{ where } \rho_0 \text{ is the explosion time (Mao, 1997). To prove that this solution is global, we need only show that } \rho_0 = \infty \text{ a.s. Define the stopping time } \rho_0 = \inf \{t \in [0, \rho_0): |y(t)| \geq k \}, k. \text{ Hence, } \rho_0 \text{ is increasing as } k \rightarrow \infty \text{ and } \rho_0 \rightarrow \infty \text{ a.s. If we can show } \rho_0 = \infty \text{ a.s., then } \tau = \infty \text{, which implies that the solution } y(t) \text{ is actually global. Let } \tilde{V}(t) := \mathbb{E} V(y(t)_{\rho(t)}, t \wedge \rho_0). \text{ By the generalized Itô formula (3) and condition (II), we obtain that for } \forall k > 0 \text{ and } t \geq 0,

$$\tilde{V}(t) = \mathbb{E} V(y_0(t)) + \mathbb{E} \int_0^t \delta V(y(s), s) \, ds$$

$$\leq \mathbb{E} V(y_0(t)) + \mathbb{E} \int_0^t \int_{-\tau}^t U_2(y(s + \tau)) d\mu_2(\tau) - \alpha U_3$$

$$\leq \int_0^t \int_{-\tau}^t U_2(y(s + \tau)) d\mu_2(\tau) + \mathbb{E} \int_0^t \int_{-\tau}^t U_3(y(s + \tau)) d\mu_2(\tau) + \int_0^t \eta(s) \, ds.$$ 

By the Fubini theorem, we get that for $i = 2, 3$,

$$\mathbb{E} \int_0^t \int_{-\tau}^t U_i(y(s + \tau)) d\mu_2(\tau) ds$$

$$= \mathbb{E} \int_0^t \int_{-\tau}^t U_i(y(s + \tau)) d\mu_2(\tau) ds$$

$$\leq \int_0^t \mathbb{E} U_i(y(s)) ds + \mathbb{E} \int_0^t \int_{-\tau}^t U_i(y(s)) d\mu_2(\tau) ds.$$ 

Note that $\alpha \geq 0$ and $U_2(x) - U_1(x) \geq -\lambda |x|^p$. Hence, substituting the inequalities above into (4) yields

$$\tilde{V}(t) \leq \mathbb{E} V(y_0(t)) + \mathbb{E} \int_0^t \int_{-\tau}^t U_2(y(s + \tau)) d\mu_2(\tau) ds + \mathbb{E} \int_0^t \int_{-\tau}^t U_3(y(s + \tau)) d\mu_2(\tau) ds$$

$$+ (-\lambda \vee \eta(t)) \mathbb{E} \int_0^t \int_{-\tau}^t |y(s + \mu_2(\tau))|^p ds + \int_0^t \eta(s) ds$$

$$\leq (1 + \int_0^t \eta(s) \, ds) C_1 + (-\lambda \vee \eta(t)) \int_0^t |y(s + \mu_2(\tau))|^p ds.$$
where $C_1 = 1 + \mathbb{E}Y(y_0, r(0)) + \mathbb{E} \int_0^t \eta(s)ds + \mathbb{E} \int_0^t U_2(y(\theta))d\theta + \beta \mathbb{E} \int_0^t U_3(y(\theta))d\theta$. By condition (I), we have that for any $s > 0$,

$$
\mathbb{C}_0 \mathbb{E}[y(s \wedge \rho')^p] \leq \hat{V}(s) + C_1 \sup_{t \in [s, t]} \mathbb{E}[y(u \wedge \rho')^p] \\
\leq \hat{V}(s) + C_1 \sup_{t \in [s, t]} \mathbb{E}[y(u \wedge \rho')^p].
$$

(7)

Note that $\rho' \geq 0$. Then we get

$$
\sup_{t \in [s, t]} \mathbb{E}[y(s \wedge \rho')^p] \leq \mathcal{C}_0(y_0) + \sup_{t \in [s, t]} \mathbb{E}[y(s \wedge \rho')^p] \\
\leq \mathcal{C}_0(y_0) + \frac{1}{C_0(t \wedge \rho')} \sup_{t \in [s, t]} \mathbb{E}[y(s \wedge \rho')^p] + \frac{C_1}{C_0(t \wedge \rho')} \sup_{t \in [s, t]} \mathbb{E}[y(s \wedge \rho')^p],
$$

(8)

which together with $C_1 < C_0$ gives $\sup_{t \in [s, t]} \mathbb{E}[y(s \wedge \rho')^p] \leq \frac{\mathcal{C}_0(t \wedge \rho')}{C_0(t \wedge \rho')} \sup_{t \in [s, t]} \mathbb{E}[y(s \wedge \rho')^p]$. Substituting the inequality above into (6) yields $\mathbb{E}[y(t \wedge \rho')^p] \leq (1 + \int_0^t \eta(s)ds)C_2 + C_1 \int_0^t \mathbb{E}[y(s \wedge \rho')^p]ds$, where $C_2 = \frac{\mathcal{C}_0(t \wedge \rho')}{C_0(t \wedge \rho')} + \frac{C_1}{C_0(t \wedge \rho')}$. Applying the Gronwall inequality produces

$$
\mathbb{E}[y(t \wedge \rho')^p] \leq (1 + \int_0^t \eta(s)ds)C_2 + C_1 \int_0^t \mathbb{E}[y(s \wedge \rho')^p]ds \leq (1 + \int_0^t \eta(s)ds)C_2 e^{C_1 t} = C_4(t).
$$

(9)

Note that $\mathbb{E}[y(t \wedge \rho')^p] \leq \mathbb{E}[y(t \wedge \rho')^p] 1_{\{\tau \leq t\}} \leq k^p \mathbb{P}(\rho \leq t)$. Hence, $k^p \mathbb{P}(\rho \leq t) \leq C_4(t)$. Then, for any $t > 0$, $\lim_{t \to \infty} k^p \mathbb{P}(\rho \leq t) = 0$, which together with the arbitrariness of $t$ implies $\mathbb{P}(\rho \leq \infty) = \infty$ a.s. Therefore, the solution $y(\cdot)$ is global. Especially, if $\int_0^\infty \eta(s)ds < \infty$ and $\lambda > 0$, then we can see that (9) is still true with $C_3 = 0$. Hence, we get from (9) for any $t > 0$, $\mathbb{E}[y(t)^p] = \mathbb{E}[\lim_{\tau \to \infty} y(t \wedge \rho')] \leq \lim_{\tau \to \infty} \mathbb{E}[y(t \wedge \rho')] \leq (1 + \int_0^\infty \eta(s)ds)C_2$, where we used Fatou’s lemma and $\rho \in \infty$. \hfill \Box

**Theorem 3.2.** Assume that there exist a functional $V \in C^{2,1}(D([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}_+; \mathbb{R}_+)$, three functions $U_1, U_2, U_3 : \mathbb{R}^n \to \mathbb{R}_+$, $\eta(t) \geq 0$ and some nonnegative constants $c_0 > c_1, c_2, c_3 \neq 0, \alpha > \beta$ such that

(1) $c_0 \mathbb{E}[y(0)^p] - c_1 \int_0^\infty \mathbb{E}[y(s)^p]ds \leq V(y, t) \leq c_2 |y|^\alpha + c_3 \int_0^\infty \mathbb{E}[y(s)^p]ds;

(II) $2\mathbb{E}[y(t)^p] \leq U_1(y(0)) + \int_0^t \mathbb{E}[U_2(y(s))]ds - \alpha U_3(y(t)) + \beta \mathbb{E} \int_0^t U_3(y(\theta))d\theta;

(III) $U_2(y) - U_1(y) \leq 0$ and there exists a $\gamma_0 > 0$ such that $\gamma_0(c_2 + c_3 \mathbb{E} \int_0^\infty y(s)^p ds) \leq (\gamma - \gamma_0)U_2(y(t)) + \eta_0 U_2(y(t)) \leq 0$ for all $x \in \mathbb{R}^n$, where $\{\eta_0\}_{i=1}^m$ are probability measures defined on $[-\tau, 0]$. Then the unique global solution to SDE (1) satisfies

(a) $\lim_{t \to \infty} \mathbb{E}[y(t)^p] \leq \frac{1}{\sup_{0 \leq r \leq \tau} \eta^{\frac{1}{p}}} \eta_0$ for any $t > 0$;

(b) $\lim_{t \to \infty} \mathbb{E}[y(t)^p] = 0$ if $\lim_{t \to \infty} \int_0^t \mathbb{E} e^{\gamma(t-s)}\eta(s)ds = 0$;

(c) $\mathbb{E}[y(t)^p] \leq C e^{-\gamma t}$ for some $C > 0$ if $\int_0^\infty e^{\gamma t} \eta(t)dt < \infty$;

where $\gamma \in (0, \gamma^*)$, $\gamma^* = \gamma_0 \wedge \gamma_1 \wedge \gamma_2$, $\gamma_1 = \frac{1}{\alpha} \log(\frac{\alpha}{\beta})$, $\gamma_2 = \frac{1}{\beta} \log(\frac{\beta}{\alpha})$.

Proof. Note that $U_2(y) - U_1(y) \leq 0$ implies $U_2(y) - U_1(y) \leq |\lambda| |x|^\beta$. Then, condition (I) holds. Based on Theorem 3.1, the solution is regular. Letting $Z^k(t) = e^{\theta(y(t))}V(y(t), \rho_k)$ and applying Ito’s formula (2), we obtain that

$$
Z^k(t) = V(y(0), 0) + \int_0^t e^{-\gamma s} \Delta V(y(s), s)g(y(s), s)dw(s)
$$

$$
+ \int_0^t e^{\gamma s}[yV(y(s), s) + 2V(y(s))]ds
$$

$$
\leq V(y(0), 0) + \int_0^t e^{-\gamma s} \Delta V(y(s), s)g(y(s), s)dw(s)
$$

$$
+ \int_0^\tau e^{\gamma s} \eta(s)ds + c_2 \gamma \int_0^\tau e^{\gamma s} |y(s)|^2ds
$$

$$
+ c_3 \gamma \int_0^\tau e^{\gamma s} \int_0^\tau |y(u + s)|^2du ds
$$

$$
- \alpha \int_0^\tau e^{\gamma s} U_2(y(s))ds - \int_0^\tau e^{\gamma s} U_2(y(s))ds
$$

$$
+ \beta \int_0^\tau e^{\gamma s} \int_0^\tau U_2(y(u + s))ds du
$$

$$
+ \int_0^\tau e^{\gamma s} \int_0^\tau U_2(y(u + s))ds du
$$

$$
= Z^k(\tau).
$$

(11)

Note that for $i = 2, 3, 4,$ and $h : \mathbb{R}^n \to \mathbb{R}_+$, by interchanging the order of integration,

$$
\int_0^\tau e^{\gamma s} \int_0^\tau h(y(u + s))du ds
$$

$$
\leq e^{\gamma \tau} \int_0^\tau e^{\gamma s} h(y(s))ds + e^{\gamma \tau} \int_0^\tau e^{\gamma s} h(y(s))ds.
$$

(12)

Substituting (12) into (11), we get

$$
Z^k(t) \leq C_1(\gamma) + \int_0^\tau e^{\gamma s} \eta(s)ds
$$

$$
+ (c_2 + c_3 \gamma) e^{\gamma \tau} \int_0^\tau e^{\gamma s} |y(s)|^2ds
$$

$$
+ (\beta e^{\gamma \tau} - \alpha) \int_0^\tau e^{\gamma s} U_2(y(s))ds
$$

$$
- \int_0^\tau e^{\gamma s} (U_2(y(s)) - e^{\gamma s} U_2(y(s)))ds.
$$

(13)
where $C_t(y) = E[V(y_0, 0)] + c_1 y e^{\tau t} E \int_0^\infty e^{\tau s} |y(s)|^p ds + \beta e^{\tau t} E \int_0^\infty e^{\tau s} U_3(y(s))ds$. From $\alpha > \beta$, we know that $\beta e^{\tau t} - \alpha < 0$ for any $y < y_1 := \frac{1}{\beta} \log \left( \frac{y_0}{\beta} \right)$. Then we get from (13) that for $y < y_1$,

$$E(Z)(t) \leq C_t(y) + \int_0^t e^{\tau s} \eta(y, t)ds,$$

where $h(y, x) = y(C_2 + c_4 e^{\tau t}|x| - \lambda + c_4(e^{\tau t} - 1))$.

Note that $h(0) = -\lambda$ and $h(y) > 0$ for $y$ satisfying $y(c_2 + c_4 e^{\tau t} - \lambda) > 0$. In addition, $h(y) > 0$ for any $y > 0$. Then, there exists a unique root $y_2$ such that $h(y_2) = 0$. Hence, $h(y) < 0$ for any $y \in (0, y_1 \wedge y_2)$. Therefore, we have from (18) that $-h(y) \int_0^\infty e^{\tau s} \eta(y, s)ds \leq C_t(y) + \int_0^\infty e^{\tau s} \eta(y, s)ds$.

By Fatou’s lemma, letting $k \to \infty$, for any $t > 0$, $-h(y) \int_0^\infty e^{\tau s} \eta(y, s)ds \leq C_t(y) + \int_0^\infty e^{\tau s} \eta(y, s)ds < \infty$, which implies the desired assertion. □

**Remark 4.2.** Theorem 2.1 in Kolmanovskii and Myshkis (1999, p. 389) asserts that if the coefficients $f, g$ satisfy the global Lipschitz condition and there exist $C_1, C_2, C_3$ such that

(i) $C_1 \|\eta(t)\|^q \leq E[V(y, t)] \leq C_2 \sup_{t \leq s < 0} E[|y(t) + s)|^2,$

(ii) $E[V(y, t) - V(y, s)] \leq -C_3 \int_s^t E[|y(u)|]^2 du$ for any $s \leq 0,$

then the solution is mean square asymptotically stable. Our stability results in Theorems 4.1 and 4.2 improve Theorem 2.1 in Kolmanovskii and Myshkis (1999) in the following four points: (1) Theorems 4.1 and 4.2 do not require the coefficients to be globally Lipschitz. (2) There is no requirement on the positive definiteness of $V(y, t)$, (3) Our results can produce the mean square exponential stability. (4) Condition (II) is broader than condition (ii) above since condition (II) can deal with the case with additive noises.

**Theorem 4.3.** Assume that there exist a functional $V \in C^{2,1} (D[\tau, 0]; R^m \times R_+)$, three functions $U_1, U_2, U_3 : R^m \to R_+$, and some nonnegative constants $c_2 C_3 \geq 0, c_4 \alpha > \beta, \lambda > 0$ such that conditions (I’), (II’), and (III’) hold. If $\int_0^\infty e^{\tau t} dt < \infty$, then SFDEs (1) admits a unique global solution and the solution satisfies that for any initial data $y_0 \in C([-\tau, 0]; R^n)$, $\limsup_{t \to \infty} \frac{1}{\beta} \log |y(t)| \leq \frac{\alpha}{\beta}$, a.s., where $y(\gamma) \in (0, \gamma^*)$ is defined in Theorem 4.2.

**Proof.** For $y > 0$, let $Z(t) = e^{\tau t} V(y, t)$. Applying the functional Itô formula to $e^{\tau t} V(y, t)$ and using (II) yield

$$Z(t) = V(y_0, 0) + \int_0^t e^{\tau s} \eta(y, t)ds - \alpha \int_0^t e^{\tau s} U_3(y(s))ds$$

$$\leq -c_2 \gamma \int_0^t e^{\tau s} (y^2 + |U_2(y)|^p + U_1(y))ds$$

$$\leq -c_3 \gamma \int_0^t e^{\tau s} U_1(y(s))ds + M(t)$$

$$\leq -c_3 \gamma \int_0^t e^{\tau s} U_1(y(s))ds + M(t)$$

$$\leq -c_3 \gamma \int_0^t e^{\tau s} U_1(y(s))ds + M(t)$$

$$\leq -c_3 \gamma \int_0^t e^{\tau s} U_1(y(s))ds + M(t)$$

$$\leq -c_3 \gamma \int_0^t e^{\tau s} U_1(y(s))ds + M(t)$$

where

$$h(y) = y(C_2 + c_4 e^{\tau t} - \lambda) + c_4(e^{\tau t} - 1).$$

(19)
where \( M(t) = \int_0^t e^{-\gamma s} \Delta_t V(y_s, s) g(y_s, s) dw(s) \). Substituting (12) into (21), we get that

\[
Z(t) \leq C_0(\gamma) + (\beta e^{\gamma t} - \alpha) \int_0^t e^{\gamma s} U_2(y(s)) ds + M(t)
+ h(\gamma) \int_0^t e^{\gamma s} |y|^3 ds + \int_0^t e^{\gamma s} \eta(s) ds,
\]

where \( C_0(\gamma) = V(y_0, 0) + c_{\gamma} e^{\gamma t} \int_0^t |y|^3 ds + e^{\gamma t} \int_0^t U_2(y(s)) ds \) and \( h(\gamma) \) is defined by (19). When \( \gamma \in (0, \gamma^*), \) then \( h(\gamma) \leq 0 \). Moreover, we have from (22) that

\[
-h(\gamma) \int_0^t e^{\gamma s} |y|^3 ds \leq Y(t),
\]

where \( Y(t) = C_0(\gamma) + (\beta e^{\gamma t} - \alpha) \int_0^t e^{\gamma s} U_2(y(s)) ds + M(t) + \int_0^t e^{\gamma s} \eta(s) ds \geq 0 \). Note that \( \beta e^{\gamma t} - \alpha < 0 \). As \( t \to \infty \), we have

\[
\int_0^t e^{\gamma s} \eta(s) ds \leq \int_0^t e^{\gamma s} |y|^3 ds < \infty
\]

and the semi-martingale convergence theorem (Mao, 1997, p. 14) implies \( \lim_{t \to \infty} Y(t) < \infty, \) a.s. This together with (23) implies \( \int_0^t e^{\gamma s} |y|^3 ds < \infty, \) a.s. Therefore, the desired assertion follows. \( \square \)

Note that the conditions of almost sure exponential stability in Theorem 4.3 are the same as that of \( \phi \) moment exponential stability in Theorem 4.1. A natural question is: Can we obtain the almost sure asymptotic stability under the conditions for the \( \phi \) moment asymptotic stability in Theorem 4.1? The following simple equation without delay shows that this may be impossible.

\[
dy(t) = -y(t) dt + \sigma(t) dw(t).
\]

It is easy to see that \( \lim_{t \to \infty} E|y(t)|^2 = 0 \) if and only if

\[
\lim_{t \to \infty} \int_0^t e^{-\sigma(t)}(\sigma(s))^2 ds = 0.
\]

From (Appleby, Cheng, & Rodkina, 2011), we know that \( \lim_{t \to \infty} E|y(t)| = 0 \) a.s. if \( \lim_{t \to \infty} \sigma(t) \log t = 0 \), and only if

\[
\liminf_{t \to \infty} \sigma(t) \log t = 0.
\]

Hence, if we choose \( \sigma(t) = \frac{1}{\log t} \) for \( t > 1 \), then (25) holds (L'Hôpital's rule), and \( \lim_{t \to \infty} E|y(t)|^2 = 0 \). But (26) is defined, and then \( E(\lim_{t \to \infty} |y(t)| = 0) < 1 \).

Remark 4.3. In the above, we developed degenerate Lyapunov functionals to investigate the asymptotic behaviors of SFDEs. The conditions (I') and (II) is required to hold for all \( \psi \in C^1([\tau, 0], \mathbb{R}^d) \times \mathbb{R}^d \). Actually, in many complex networked systems, we only require these conditions to be true on the solution path. Moreover, based on the solution paths, the structure properties of the complex systems might be helpful for the asymptotic analysis (see the multi-agent application in the subsequent section).

4.2. Stochastic stability of semi-linear SFDEs

Consider the following semi-linear SFDE,

\[
dy(t) = \left[ \sum_{i=1}^d A_i y(t - \tau) + f(y_t, t) \right] dt + \sum_{i=1}^{d} g_i(y_t, t) dw_i(t),
\]

where \( y_t = \{y(t + \theta) : \theta \in [-\tau, 0] \}, \) \( A_i \in \mathbb{R}^{n \times n}, \) \( \tau > 0, \) \( 0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_d \) are delays, \( f \) and \( g_i \) satisfy the following linear growth condition.

**Assumption 4.1.** For each positive definite matrix \( P \) there exist matrices \( D_{1P} \geq 0 \) and \( D_{2P} \geq 0 \) such that for all \( t \geq 0 \) and

\[
\varphi \in C([-\tau, 0], \mathbb{R}^n),
\]

\[
f^{T}(\varphi, t) P f(\varphi, t) = q^2 \int_{-\tau}^{0} \varphi^{T}(s) D_{1P} \varphi(s) d\tilde{\mu}(s),
\]

\[
\sum_{i=1}^{d} \bar{g}_i^{T}(\varphi, t) P g_i(\varphi, t) = \int_{-\tau}^{0} \varphi^{T}(s) D_{2P} \varphi(s) d\tilde{\mu}(s),
\]

where \( q \geq 0, \tilde{\mu} \) is a probability measure on \([0, -\tau] \).

The delay-independent and delay-dependent stability has been discussed in the important works (Kolmanovskii & Myshkis, 1999; Kolmanovskii & Shaikhet, 1997, 1998; Kovalev, Kolmanovskii, & Shaikhet, 1998) for linear systems. In this section, we consider the delay-dependent stability for semi-linear systems, where without loss of generality, we assume that \( \sum_{i=0}^{d} A_i \) is Hurwitz if and only if \( l = d \), which is consistent with Theorem 5.3 in Kolmanovskii and Myshkis (1999) with \( m_0 = m \).

**Theorem 4.4.** Suppose that Assumption 4.1 holds. Let \( \bar{A} = \sum_{i=0}^{d} A_i \). Then stochastic system (27) is mean square and almost surely exponentially stable if there exists a matrix \( P > 0 \) such that

\[
S := \bar{A}^{T} P + P \bar{A} + \sum_{i=1}^{d} \tau_i \bar{A}^{T} P A_i + (1 + q) \sum_{i=1}^{d} A_i^{T} P A_i
+ q P + q(1 + \sum_{i=1}^{d} \tau_i) D_{1P} + D_{2P} < 0.
\]

**Proof.** Let \( P \) be fixed. Define

\[
V_{1}(\varphi, t) = V_{1}(\varphi(t)) + V_{2}(\varphi, \varphi) \in C([-\tau, 0], \mathbb{R}^n),
\]

with \( V_{1}(\varphi(t)) = \langle \varphi(0), \sum_{i=1}^{d} \int_{-\tau_i}^{0} A_i \psi(s) ds \rangle + \sum_{i=1}^{d} \int_{-\tau_i}^{0} \psi^{T}(s) D_{1P} \psi(s) ds \) and \( V_{2}(\varphi, \varphi) = \langle 1 + q, \sum_{i=1}^{d} \int_{-\tau_i}^{0} \varphi^{T}(s) A_i^{T} P A_i \psi(s) ds \rangle \). It is easy to see that condition (I') holds. It can be deduced that

\[
\begin{align*}
\mathcal{L} V_{1}(\varphi, t) &= \psi^{T}(0) A_i^{T} P + P \bar{A} \psi(0) \\
&+ 2 \psi^{T}(0) P f(\varphi, t) + \sum_{i=1}^{d} \bar{g}_i^{T}(\varphi, t) P g_i(\varphi, t) \\
&+ \int_{-\tau}^{0} 2 \psi^{T}(s) \varphi^{T}(s) P A_i\psi(s) ds \\
&+ 2 \int_{-\tau}^{0} \bar{A}_{j} \psi(s) ds \\
&+ q f^{T}(0) P f(\varphi, t) + q \int_{-\tau}^{0} \varphi^{T}(s) D_{1P} \varphi(s) d\tilde{\mu}(s) + 2 \int_{-\tau}^{0} \psi^{T}(s) D_{2P} \psi(s) d\tilde{\mu}(s).
\end{align*}
\]

By the elementary inequality \( x^{T} y + y^{T} x \leq x^{T} \alpha x + y^{T} (\alpha - 1) y \), for any positive definite matrix \( \alpha > 0 \), and \( x, y \in \mathbb{R}^{n} \), we get

\[
\begin{align*}
\mathcal{L} V_{1}(\varphi, t) &\leq \varphi^{T}(0) \psi(0) + \int_{-\tau}^{0} \varphi^{T}(s) D_{1P} \varphi(s) d\tilde{\mu}(s) \\
&+ q(1 + \sum_{i=1}^{d} \tau_i) \int_{-\tau}^{0} \varphi^{T}(s) D_{2P} \varphi(s) d\tilde{\mu}(s).
\end{align*}
\]
where \( S_1 = \overline{A}^TP + PA + \sum_{i=1}^{d} \tau_i \overline{A}^T P \overline{A} + qP \). By the definition of \( V_2(\cdot, t) \), we have \( \mathcal{L}V_2(\cdot, t) = (1 + q) \sum_{i=1}^{d} \tau_i \overline{A}^T P \overline{A} + \sum_{i=1}^{d} \tau_i \overline{A}^T P A_i + \sum_{i=1}^{d} \tau_i A_i P A_i + qP \). This together with the estimations of \( \mathcal{L}V_1(\cdot, t) \) gives
\[
3 \mathcal{L}V(\cdot, t) \leq \mathcal{L} \left( \mathcal{L} \right) V(\cdot, t) \leq \alpha \mathcal{L} V(\cdot, t),
\]
where \( \alpha = \frac{\alpha_1^2 + \overline{\alpha} + \sum_{i=1}^{d} \tau_i \overline{A}^T P \overline{A} + 1 + q \sum_{i=1}^{d} \tau_i \overline{A}^T P A_i + \overline{A}^T P A + qP}{\alpha_1^2 + \overline{\alpha} + \sum_{i=1}^{d} \tau_i \overline{A}^T P \overline{A} + \sum_{i=1}^{d} \tau_i A_i P A_i + \sum_{i=1}^{d} \tau_i A_i P A_i + qP} < 1 \). Hence, we get
\[
\mathcal{L}V(\cdot, t) \leq \alpha \mathcal{L} V(\cdot, t),
\]
where \( \alpha = \frac{\alpha_1^2 + \overline{\alpha} + \sum_{i=1}^{d} \tau_i \overline{A}^T P \overline{A} + 1 + q \sum_{i=1}^{d} \tau_i \overline{A}^T P A_i + \overline{A}^T P A + qP}{\alpha_1^2 + \overline{\alpha} + \sum_{i=1}^{d} \tau_i \overline{A}^T P \overline{A} + \sum_{i=1}^{d} \tau_i A_i P A_i + \sum_{i=1}^{d} \tau_i A_i P A_i + qP} < 1 \).

**Theorem 4.6.** For any given \( \bar{\tau} \geq 0 \), the trivial solution to (33) is mean square and almost surely exponentially stable if
\[
2 \overline{\alpha} \tau < 2 \alpha - \overline{\alpha}^2, \quad \overline{\alpha} \bar{\tau} \leq 2 \bar{\beta} - \alpha_1^2.
\]

**Proof.** Choose the degenerate Lyapunov functional
\[
\mathcal{V}(\cdot, t) = \| \phi(0) - \overline{\alpha} \int_{-ar{\tau}}^{0} \mathcal{L} \mathcal{V}(\cdot, s) ds \|^2.
\]
We obtain \( \mathcal{L} \mathcal{V}(\cdot, t) \leq -2 \bar{\alpha} \mathcal{V}(\cdot, t) = \sum_{i=1}^{d} \mathcal{V}(\cdot, t) \mathcal{V}(\cdot, t) + 2 \bar{\alpha} \mathcal{V}(\cdot, t) \mathcal{V}(\cdot, t) + \sum_{i=1}^{d} \mathcal{V}(\cdot, t) \mathcal{V}(\cdot, t) + \alpha_1^2 \mathcal{V}(\cdot, t) \mathcal{V}(\cdot, t). \)

By the elementary inequality \( \alpha^2 b^2 < \frac{1}{2} \alpha^2 b^2 + \frac{1}{2} \alpha^2 b^2 \), we have
\[
\mathcal{V}(\cdot, t) \leq \left( \frac{\alpha_1^2}{2} + \overline{\alpha} \right) \mathcal{V}(\cdot, t) + \left( \frac{\alpha_1^2}{2} + \overline{\alpha} \right) \mathcal{V}(\cdot, t) + \alpha_1^2 \mathcal{V}(\cdot, t) \mathcal{V}(\cdot, t).
\]
We have investigated the asymptotic stability and exponential stability of SFDEs. This is different the \( l^2 \)-stability of the random

5. Applications to multi-agent systems with nonlinear dynamics and noises

In this section, we apply our main findings to multi-agent systems with distributed random delays and establish convergence and consentability of such systems that are difficult to treat when multiple or distributed delays are encountered. Wireless communication systems are essential part of networked systems and introduce significant impact on control design and system performance. In Xu, Wang, Yin, and Zhang (2014), coordinated control and communication design was studied in which TCP-based (Transmission Control Protocol) communication protocols were employed. The main consequence of the TCP channel uncertainties is random signal transmission delays.

Inter-vehicle communications use wireless networks and are subject to signal fading, delays in each hub’s queues, re-transmission, scheduling policies in interference avoidance strategies, resulting in combined latency which is random, time varying, unpredictable, and distributed (Neely & Modiano, 2005). In our recent paper (Wang, Syed, Yin, Pandya, & Zhang, 2014), a weighted and constrained consensus control method was introduced to achieve platoon formation and robustness against communication channel uncertainties. However, distributed random delays could not be treated. Applying the new results of this paper, convergence to a desired platoon formation can now be achieved under more realistic communication latency conditions.

To proceed, we study consentability of nonlinear multi-agent systems consisting of $N$ agents. Each agent is modeled by a dynamic system

\[ \dot{y}_i(t) = f(y_i(t), y_j(t - \tau), t) + u_i(t), \quad i = 1, \ldots, N \tag{39} \]

where $y_i(t) \in \mathbb{R}^n$ is the state of the $i$th agent, $\tau$ is a delay, $u = [u_1^T, \ldots, u_N^T]^T$ is the control to be designed, $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear function satisfying the following assumption.

**Assumption 5.1.** $f(0, 0, t) = 0$ and there exists a positive constant $q$ (Lipschitz constant) such that for all $x, y, \tilde{x}, \tilde{y}, \psi(x, y, t) - f(\tilde{x}, \tilde{y}, t) \leq q(|x - \tilde{x}| + |y - \tilde{y}|)$.

The information flow structures among different agents are modeled as a directed graph $G = (V, E, A)$, where $V = \{1, 2, \ldots, N\}$ is the set of nodes with $i$ representing the $i$th agent, $E$ denotes the set of directed edges and $A = \{a_{ij}\} \in \mathbb{R}^{N \times N}$ is the adjacency matrix of $G$ with element $a_{ij} = 1$ if $i$ and $j$ are neighbors, $a_{ij} = 0$ if not. The Laplacian matrix $L$ is defined as $L = D - A$, where $D = \text{diag}(d_1, \ldots, d_N)$.

The Laplacian matrix of a directed graph $G$ is defined as \( L = D - A \), where $D = \text{diag}(d_1, \ldots, d_N)$. It is obvious that $L$ admits a zero eigenvalue, denoted by $\lambda_1$. If the digraph $G$ is balanced, then \( \tilde{L} = \frac{c^T}{c}L \) is the Laplacian matrix of the mirror digraph $G^*$ of $G$ (Olfati-Saber & Murray, 2004).

Consider the control

\[ u_i(t) = K \sum_{j \in N_i} z_{ij}(t), \tag{40} \]

where the control gain $K \in \mathbb{R}^{n \times n}$ is symmetric, $z_{ij}(t) = y_i(t - \tau_j) - y_j(t - \tau_i) + g_0(\psi(t - \tau_j) - \psi(t - \tau_i))$ denotes the measurement of the agent $i$ from its neighbor agent $j$, $\tau_i$ and $\tau_j$ are delays, $g_0(t)$ is a scalar independent Gaussian white noise, and $g_0 \in \mathbb{R}$ is the noise intensity factor. Here, we make the following assumptions.

**Assumption 5.2.** The noise processes \( \{g_0(t) \mid i, j = 1, 2, \ldots, N\} \) satisfy \( \int_0^t g_0(s)ds = w(t) \) for $t \geq 0$, where \( \{w_i(t), i, j = 1, 2, \ldots, N\} \) are independent Brownian motions.

**Assumption 5.3.** For each \( (i, j) \), $g_0(0) = 0$ and there exists a positive constant $\sigma$ such that $|g_0(x) - g_0(y)| \leq \sigma|x - y|$, for all $x, y \in \mathbb{R}^n$.

For the measurement of the communication delays, one can refer to (Zong, Li, Yin et al., 2017, Zong, Li, & Zhang 2017). We denote the collection of all admissible distributed protocols by $\mathcal{U} = \{u(t)|u_i(t)\}$ given by (40) for all symmetric $K, \tau \geq 0$, and $i = 1, \ldots, N$.

**Definition 5.1.** The multi-agent systems (39) are said to be mean square (or almost surely) consentable with respect to (w.r.t.) $\mathcal{U}$, if there exists a protocol $u \in \mathcal{U}$ solving the mean square (or almost sure) consensus, that is, for any initial data $\psi \in \mathbb{C}([-\tau \vee \tau_2], 0)$, $\mathbb{R}^{Nn}$ and all distinct $i, j \in V$, limit $\lim_{t \to \infty} E[y_i(t) - y_j(t)] = 0$ (or limit $\lim_{t \to \infty} |y_i(t) - y_j(t)| = 0$, a.s.).

**Lemma 5.1.** For the Laplacian matrix $L$ related to a balanced graph $G$, there exists a matrix $Q \in \mathbb{R}^{N \times (N-1)}$ such that the matrix $Q = \left( \frac{1}{\sqrt{N}} \mathbf{1}_N \right)^T$ is orthogonal and

\[ Q^{-1} = \left[ \frac{1}{\sqrt{N}} \right] Q^T. \tag{41} \]

The proof of the lemma can be found in Zong, Li, and Zhang (2018). We proceed to obtain the following result.

**Theorem 5.2.** Suppose that Assumptions 5.1, 5.2, and 5.3 hold and the graph $G$ is balanced. If

\[ 4q(\lambda_2(\tilde{L}) - \frac{1}{N}) - \sigma^2 + (1 + q)\lambda(2 + \tau_1) < \lambda_2^2(\tilde{L}), \tag{42} \]

then multi-agent systems are mean square and almost sure consentable w.r.t. $\mathcal{U}$. Moreover, the mean square and almost sure consensus can be solved by the protocol (40) with $K = kl_s$, satisfying $k \in (0, k_0)$, $k_0 := \min_{\lambda \geq \lambda_1} \lambda(\tilde{L}), \tag{40} \]

where $\lambda_a = \lambda_2^2(\tilde{L}) - 4q(2 + \tau_1), \tag{42} \]

Proof. It is enough to prove that the protocol with $k$ satisfying $k \in (0, k_0)$ can solve the mean square and almost sure consensus. Note that $k$ and $k_0$ are well-defined since condition (42) holds. Let $y(t) = [y_1(t), \ldots, y_N(t)]^T$, $F(t) = [f_1(t), \ldots, f_N(t)]^T$, $\delta(t) = [\delta_1(t), \delta_2(t)]^T$, and $\delta(t) = [\delta_1(t), \delta_2(t)]^T$, where $\delta_1(t) \in \mathbb{R}^{N \times n}$.

Substituting the protocol (40) into the system (39) and using Assumption 5.2 yields $d\{\delta(t)\} = -k[L(\tilde{L}) - \tilde{L}]\delta(t) dt + F(t) dt + dF(t)$, where $\delta(t) = \lambda_1 \delta(t) dt + F(t) dt + dF(t)$, where $\delta(t) = \lambda_1 \delta(t) dt + F(t) dt + dF(t)$, where $\delta(t) = \lambda_1 \delta(t) dt + F(t) dt + dF(t)$, where $\delta(t) = \lambda_1 \delta(t) dt + F(t) dt + dF(t)$.

Write the above as $d\delta(t) = \delta_1(t) + \delta_2(t) + \delta_3(t)$. Define $\delta_1(t) = \lambda_1 \delta(t) dt + F(t) dt + dF(t)$, where $\delta_1(t) = \lambda_1 \delta(t) dt + F(t) dt + dF(t)$.

Then, by the definition of $Q$ given in Lemma 5.1, we have $\tilde{\delta}(t) = \frac{1}{\sqrt{N}} \mathbf{1}_N^T \delta(t) dt + k[L(\tilde{L}) - \tilde{L}] \delta(t) dt + dM(t)$, where $\delta(t) = \frac{1}{\sqrt{N}} \mathbf{1}_N^T \delta(t) dt + k[L(\tilde{L}) - \tilde{L}] \delta(t) dt + dM(t)$, where $\delta(t) = \frac{1}{\sqrt{N}} \mathbf{1}_N^T \delta(t) dt + k[L(\tilde{L}) - \tilde{L}] \delta(t) dt + dM(t)$.
Define a new degenerate Lyapunov functional

\[ V(\bar{\delta}(t), t) = V_1(\bar{\delta}(0)) + k^2(2q + 1)|\bar{\delta}(t) - \delta(t)|^2 - \int_{t-\tau_1}^t \int_{\delta(s)}^0 (\bar{\delta}(u) - \delta(s))^2 \, du \, ds + \frac{q}{2} \int_{t-\tau_1}^t (|\delta(s)|^2 + q \bar{\delta}(t - \tau_1))^2 \, ds \]

We obtain that

\[ \Delta V(\bar{\delta}(t), t) \leq -(k^2 - N - \frac{1}{N}) \sigma^2 \Phi(t) \]

\[ + 2((1 + q)k^2|\bar{\delta}(t)|^2 + 2q + qr_1)|\bar{\delta}(t)|^2 \]  

(49)

Moreover, due to the balanced graph, we have \( \Phi(t) = \delta^2(t)|C^2 + L + \lambda | I_q|\delta(t)| \geq 2\lambda_2(\bar{\delta}(t))|I_q|\delta(t)|. \) From (49), we get \( \Delta V(\bar{\delta}(t), t) \leq 2h(t)|\delta(t)|^2, \) where \( h(t) = ak^2 - \lambda_2(I)|k| + q|2 + r_1|a = N - \frac{1}{N} \sigma^2 \lambda_2(I) + (1 + q)|\bar{\delta}(t)|^2. \) Moreover, it is easy to see that \( h(t) < 0 \) for \( t \geq (k, k). \)

Therefore, Theorem 4.2 and the linear growth condition of \( f(\cdot, t) \) and \( g_i(\cdot) \) imply the mean square and almost sure exponential stability of the solution to (43), that is, \( \mathbb{E}|\delta(t)|^2 \leq C\rho^{-\alpha(t)} \), \( \limsup_{\rho \to 0} \mathbb{E} |\delta(t)|^2 < \frac{M}{\rho}, \) a.s. for certain \( C_0, \gamma > 0. \) Note that \( \delta(t) = y_1 - \frac{1}{N} \sum_{i=1}^N y_i(t), i = 1, \ldots, N. \) Then for \( i \neq j, |y_j(t) - y_i(t)| \leq |\delta_i(t)| + |\delta_i(t)|. \) The desired assertion follows. \( \square \)

**Theorem 5.3.** Suppose that Assumptions 5.1–5.3 hold, that the graph \( \mathcal{G} \) is balanced, that \( \tau_1 = 0, \) and that

\[ 8N - \frac{1}{N} \sigma^2 < \lambda_2(I). \]

Then multi-agent systems are mean square and almost surely consensusable w.r.t. \( i. \) Moreover, the mean square and a.s. consensus can be solved by protocol (40) with \( K = k_I \) satisfying \( k < (k, k) = \lambda_2(I)|k|^{-\frac{1}{2}} \), \( \tilde{k} = \lambda_2(I)|k|^{-\frac{1}{2}} \). \( a = \lambda_2(I)|k|^{-\frac{1}{2}}. \)

**Remark 5.1.** Theorems 5.2 and 5.3 show that if condition (42) or condition (50) holds, then for any delay \( \tau \) in the nonlinear dynamics and delay \( \tau_2 \) in noise terms, the mean square and almost sure consensus can be achieved.

6. Concluding remarks

This work established the stochastic stability of SFDEs by using degenerate Lyapunov functionals under local Lipschitz conditions. The analysis of delay-dependent stability shows that pure delay terms in SFDEs can dominate the asymptotic behavior of SFDEs. Most importantly, these stability criteria enable us to design delayed state-feedback controllers for networked systems. Note that the degenerate Lyapunov functionals may also be used to examine the ergodicity of SFDEs whose coefficients may not satisfy the classical one-sided Lipschitz condition in Bao, Yin, and Yuan (2014), and pure delay terms may dominate the existence of stationary distributions. For another extension, the degenerate Lyapunov functionals can also be applied to examine SFDEs with Poisson jumps, neutral SFDEs, and their stabilization problems. In addition, this paper established the consentability for first-order multi-agent systems. We hope that the stability theorems can be applied to the heterogeneous multi-agent systems, and their leader-following versions.

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Appendix. Proof of Theorem 2.1

With the motivation of Cont and Fournié (2013), an outline of the proof is given below. Let the interval $[t−r, t]$, be fixed. Without loss of generality, we use the uniform partition of $[t−r, t]$: $\{t_0, t_1, \ldots, t_k\}$ for any continuous semi-martingale $y_t$, $y_t$ is uniformly continuous on $[t−r, t]$. Then $h_{t_k} \rightarrow 0$ as $\Delta \rightarrow 0$. Define $\tilde{y}_k(\theta) = \sum_{l=t_k}^{t_{k+1}} y(t_k) 1_{(t_k, t_{k+1})}(\theta) + y(t_k) 1_{(0, t_{k})}(\theta)$, $k = 0, \ldots, N$, which is an approximation for function $y_{t_k}$. Consider the decomposition

$$V(\tilde{y}_{t_k}, t_k) + V(\tilde{y}_{t_{k+1}}, t_{k+1}) - V(\tilde{y}_{t_{k+1}}, t_k) = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \frac{1}{2} \text{tr}(\nabla_y V(\tilde{y}_{t_{n-1}}, t_n) 1_{[t_{n-1}, t_n]}(s))ds \, y(s)ds.$$

Let $h_k = \tilde{y}(t_{k+1}) - \tilde{y}(t_k)$ and $P(h) = V(h_{t_k}, t_{k+1})$. Then, $P(h_k) = P(0)$ and $P(0) = V(h_{t_k}, t_{k+1}) - V(h_{t_{k+1}}, t_k)$. Since $V \in C^{2,1}(\mathbb{R}^{N} \times \mathbb{R}^{N})$, $P$ is a $C^2$ function with $P''(h) = \nabla_y V^T h_{t_{k+1}} + 1_{[t_k, t_{k+1}]}(s)$ and note that $\tilde{y}_{t_{k+1}} = \tilde{y}_{t_{k}}$ is a measurable function. Applying the Itô formula yields that for $\delta < \Delta$, $P(y(t_k + \delta) - y(t_k))$.

$$\begin{align*}
\text{P}(0) & = \int_{t_k}^{t_{k+1}} \nabla_y V(\tilde{y}_{t_{k+1}} - y(t_{k+1}))ds + \int_{t_k}^{t_{k+1}} \frac{1}{2} \text{tr}(\nabla_y V(\tilde{y}_{t_{k+1}} - y(t_{k+1}))ds) \, y(s)ds.
\end{align*}
$$

Hence, taking sum form $i = 0$ to $i = N - 1$ in (A.1) yields

$$V(\tilde{y}_{t_k}, t_k) = \int_{0}^{\tilde{y}_{t_k}} V(\tilde{y}_{t_k}, t_k)ds + \int_{\tilde{y}_{t_k}}^{\tilde{y}_{t_{k+1}}} V(\tilde{y}_{t_{k+1}}, t_k)(\tilde{y}_{t_k}, t_k) \, y(s)ds.$$ 

References


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